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Office Note No. 3

Joint Numerical Weather Prediction Unit

Two Computational Difficulties Fundamental to the Balance Equation

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August 31, 1956

The purpose of this note is to point up two fundamental difficulties in the solution of the balance equation, and the dangers inherent in ignoring them. Consider the balance equation in its form most familiar to us.

$$f(\psi_{xx}+\psi_{yy})-2\psi_{xy}^2+2\psi_{xx}\psi_{yy}+\psi_x f_x+\psi_y f_y=\phi_{xx}+\phi_{yy} \qquad (1)$$

If a rectangular net is adopted, of mesh length h, a finite differences transformation of equation (1) is

$$\begin{aligned}
&\frac{f_{i,j}}{h^2}\left(\psi_{i-1,j}+\psi_{i,j-1}+\psi_{i+1,j}+\psi_{i,j+1}-4\psi_{i,j}\right)\\
&-\frac{1}{8h^4}\left(\psi_{i-1,j+1}-\psi_{i-1,j-1}+\psi_{i+1,j-1}-\psi_{i+1,j+1}\right)^2\\
&+\frac{2}{h^4}\left(\psi_{i-1,j}+\psi_{i+1,j}-2\psi_{i,j}\right)\left(\psi_{i,j-1}+\psi_{i,j+1}-2\psi_{i,j}\right)\\
&+\frac{1}{4h^2}\left(\psi_{i+1,j}-\psi_{i-1,j}\right)\left(f_{i+1,j}-f_{i-1,j}\right)\\
&+\frac{1}{4h^2}\left(\psi_{i,j+1}-\psi_{i,j-1}\right)\left(f_{i,j+1}-f_{i,j-1}\right)\\
&=\frac{1}{h^2}\left(\phi_{i-1,j}+\phi_{i,j-1}+\phi_{i+1,j}+\phi_{i,j+1}-4\phi_{i,j}\right)
\end{aligned} \qquad (2)$$

Let us discuss the solution of the system (2), without regard to the correspondence of such solution with the solution of the differential equation (1). With values of ψ given on the boundary of the net, the system (2) represents a set of N simultaneous algebraic equations in N unknowns, where N is the number of non-boundary points in the net, and the value of ψ at each non-boundary point is one of the unknowns. Each of the set of N equations is quadratic, so that without further restriction on their solution, the set does not have a unique solution, but rather 2^N independent solutions. One fundamental difficulty is the determination of which of the 2^N solutions we want, and the restrictions necessary during the solving-process to obtain that solution.

The other fundamental difficulty also arises from the non-linear character of the set. As a general set, the system (2) does not have solutions confined to real numbers. The solutions in general have imaginary components, which have no meaning in the physical problem we are dealing with. Let us re-arrange equation (2).

$$\frac{1}{2}\left(\frac{\psi_{i-1,j}+\psi_{i,j-1}+\psi_{i+1,j}+\psi_{i,j+1}-4\psi_{i,j}}{h^2}+f_{i,j}\right)^2$$
$$=\frac{1}{2}\left(\frac{\psi_{i-1,j+1}-\psi_{i-1,j-1}+\psi_{i+1,j-1}-\psi_{i+1,j+1}}{2h^2}\right)^2$$
$$+\frac{1}{2}\left(\frac{\psi_{i-1,j}-\psi_{i,j+1}+\psi_{i+1,j}-\psi_{i,j-1}}{h^2}\right)^2 \qquad (3)$$
$$+\left(\frac{\phi_{i-1,j}+\phi_{i,j-1}+\phi_{i+1,j}+\phi_{i,j+1}-4\phi_{i,j}}{h^2}+\frac{1}{2}f_{i,j}^2\right)$$
$$-\left(\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2h}\cdot\frac{f_{i+1,j}-f_{i-1,j}}{2h}\right)$$
$$-\left(\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2h}\cdot\frac{f_{i,j+1}-f_{i,j-1}}{2h}\right)$$

With regard to the possibility of complex solutions, it is apparent that, if in the solution the right hand side of equation (3) is anywhere negative, the solution must have imaginary components. If an iterative method of solution were applied which was designed for cases with real solutions only, one should expect not only non-convergence in the case of complex solutions, but divergence at an exponential rate. Thus it would seem wise at least to form the right-hand side of equation (3) continually during the iteration, check its sign, and stop the computation if it becomes negative. This minimum procedure is obviously not adequate for an operational program. In the case of an operational program, one could restrict the right-hand side of (3) by artificially restricting the third term to sufficiently large positive values. This is the procedure which has been adopted in Stanley Herman's program.

How may the problem of choosing among the 2^N independent solutions be treated? It has been shown by the writer (see reference) that if one applies the Liebmann process in a straight-forward manner, he is faced at each point during each sweep with a choice between two roots of a quadratic equation. One of the roots implies a positive absolute vorticity(the quantity which is squared on the left-hand side of equation (3)), the other a negative absolute vorticity. Oné could assign a combination of positive and negative signs of absolute vorticity to the collection of non-boundary points, and presumably arrive at a solution for any of the combinations. If this be true, the family of solutions for the various combinations of signs must be the 2^N solutions of the set of quadratic equations, for there are 2^N such combinations. Thus, the 2^N solutions of the set of simultaneous quadratic equations represented by equations (2) or (3), must correspond to the 2^N combinations of algebraic sign of absolute vorticity at the N non-boundary points. Meteorologists will see a clear choice from a mong the 2^N solutions--their choice will be that solution which has positive absolute vorticity at each non-boundary point.

It should be apparent from the preceding discussion that a program for solving the balance equation can be considered adequate only if it insures that absolute vorticity is positive everywhere in the solution.

Louis P. Carstensen has recently shown that a "cyclic" procedure converges to a solution. So far, he has applied his technique to only two cases, but there is no reason to disbelieve its generality, so long as no cases are encountered with imaginary components in their solution. Briefly, the current application of his procedure involves solving the set (2) with all terms but the first estimated from the result of the previous cycle. After arriving at a solution, he re-estimates all terms except the first, and again comes to a solution, and so on. Carstensen's cyclic process, as a general procedure, is the most powerful method we have to solve the balance equation, for reasons of speed and general efficiency. Undoubtedly, it can be applied with equal success to other non-linear differential equations. Convergence criteria have not as yet been worked out, but this problem will undoubtedly yield to analysis.

The current application of Carstensen's method to equation (2), however, ignores the difficulties discussed here, and modification of the current program to take them into account would be a major task, indeed. In its present form, the program has the latent dangers of (1) divergence when a case with a complex solution is encountered, and (2) convergence to an improper solution--a solution in which the absolute vorticity is negative at some points. So long as the program is run in conjunction with Herman's program, these dangers are not very serious, but at best, if they are realized, they would lead to a messy operation at the machine. I submit, however, that our aim now should be eventually to dispose of the old-type program, as discussed in the reference, and to write future balance equation programs with Carstensen's method. This can be done safely, only if the provisions against the two difficulties are incorporated along with his method--and also, only if we have some assurance that Carstensen's method is truly convergent for all cases we are likely to encounter. The latter safeguard can be realized by running the Carstensen method in conjunction with Herman's program for the remainder of our 701-days. If Herman's program consistently takes only one pass of the field, we could safely conclude that his program is not necessary.

As pointed out previously, provisions against the two difficulties, which are the subject of this paper, cannot be practically incorporated into the present program of the Carstensen method. It is proposed here that a new program be written, applying the Carstensen method to equation (3). In the new program, the terms on the right-hand side would be estimated from the solutions of the previous cycle, their square root would be taken, and each cycle would then involve the solution of a Poisson equation. Incorporation of safeguards into such a program would be an easy matter, indeed.

REFERENCE

Shuman, F. G., 1955: A method for solving the balance equation, Technical Memorandum No. 6, Joint Numerical Weather Prediction Unit, 23 May 1955, 12 pp.